Lecture 20

Cohomology. $P \subset G$ parabs $\underset{\text{standardize}}{\rightsquigarrow}$ $G/P \cong G/P_\Theta$, $\overset{W_P}{\overset{\|}{W_\Theta}} \subset W$

$H_{2k}(G/P) \cong \mathbb{Z}^{r_k(W/W_P)}$ $\quad r_k = \#$ elements of length k.

In W/W_P, length means the length of the shortest element.

Case $P = B$ $\Theta = \phi$ recovers G/B.

BTW: BB gives algo to get a word representing w_0!

Univ coef: $H^{2k}(G/P) \cong \mathrm{Hom}(H_{2k}, \mathbb{Z}) \cong \mathbb{Z}^{r_k(W/W_P)}$ and odd ones zero

(No Ext^1 term as all homology free) $\quad 0 \to \mathrm{Ext}^1 \to H^\bullet \to \mathrm{Hom} \to 0$

$\{\sigma \in W/W_P \mid \ell(\sigma) = k\}$ give basis for H_{2k} (cellular)

G/P is a compact oriented mfld of real dim $2N$, $N = \dim_{\mathbb{C}} = \ell(w_0 W)$

($\mathbb{C}$ str gives orientation: declare that if basis has form $v_1, Jv_1, v_2, Jv_2, \dots v_N, Jv_N$ then it is positive)

Poincaré duality has several formulations.

1) $H_i(G/P) \cong H^{2N-i}(G/P)$

2) $\exists$ nondegen pairing $H^i(G/P) \times H^{2N-i}(G/P) \to \mathbb{Z}$

why?

① Class $c \overset{\text{UCT}}{\rightsquigarrow}$ function $c^*: H^i \to \mathbb{Z}$

Now look for a class $a \in H^{2N-i}$ so the fn c^* is just cup with a

$$c^*(b) = a \cup b \in H^{2N}(G/P) \underset{\text{orient.}}{\cong} \mathbb{Z}$$

② in de Rham it's just $[\alpha], [\beta] \longmapsto \int_{G/P} \alpha \wedge \beta$.

or in general it's the cup prod.

So the cell C_σ for $\sigma \in W/W_P$ gives a homology class (the fund class of X_σ) and dually an elt

$$[X_\sigma] \in H^{2N-2\ell(\sigma)}(G/P).$$

e.g. $[X_e]$ = orient class

$[X_{w_0}] \in H^0(G/P)$ counting fn.

<u>Problem</u>. How to write $[X_\sigma][X_\tau]$ as a linear combo $[X_\eta]$?

That is, how to understand the multiplicative str of H^*.

<u>Special case</u>. If $\ell(\sigma)+\ell(\tau)=N$ then $[X_\sigma][X_\tau] = j\,[X_e]$ and then we say $\langle [X_\sigma], [X_\tau] \rangle = j$

This is sometimes called the $\overset{H^{2i}\quad H^{2N-2i}}{\text{int pairing}}$, though the same name is also used for the associated homology object.

Thm. Let $\overset{\dim a}{A}, \overset{\dim b}{B}$ be cpt submfld of cpt oriented mfld M. Let $[A], [B]$ denote the cohomology classes associated to fund classes of A, B. ($[A] \in H^{\dim M - a}$ etc.)

If $a+b = \dim_{\mathbb{R}}(M)$ and if $T_pM \cong T_pA \oplus T_pB$ $\forall p \in A \cap B$, then

$$\langle [A], [B] \rangle = \sum_{p \in A\cap B} \varepsilon_p$$

where $\varepsilon_p \in \pm 1$. If M, A, B cplx then $\varepsilon_p = 1$ $\forall p$

Now $[X_\sigma], [X_\tau]$ are not fund classes of manifolds.

But (see e.g. Hartshorne chap III) if $X \cap Y$ is contained in smooth part and int are transverse, then $\langle [X],[Y]\rangle = \# X\cap Y$.

Thm (Kleiman) Let X_σ, X_τ be Schubert varieties of comp dim. $\exists$ open dense $\mathcal{U}\subset G$ s.t. for $g\in\mathcal{U}$,

$$gX_\sigma \cap X_\tau = gC_\sigma \cap C_\tau$$

and all pants of int are transverse.

Cor. $\langle [X_\sigma],[X_\tau]\rangle$ = typical cardinality of $gC_\sigma \cap gC_\tau$.

Thm. (G/B case) Let $\sigma,\tau\in W$ s.t. $\ell(\sigma)+\ell(\tau)=\ell(w_0)$

Then if $\sigma\neq w_0\tau$, $\exists$ $g\in G$ s.t. $gX_\sigma\cap X_\tau=\phi$.

If $\sigma=w_0\tau$, then $\exists g\in G$ s.t. $gX_\sigma\cap X_\tau$ is a single point that lies in $gC_\sigma\cap C_\tau$, where they meet transversely.

Cor. $\langle [X_\sigma],[X_\tau]\rangle = \begin{cases} 1 & \sigma\tau^{-1}=w_0 \\ 0 & \text{else}\end{cases}$

Pf idea. Use g representing w_0. $w_0Bw_0=B^-$ opposite.

$\left(\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha \oplus \mathfrak{h}\right)$

Just as $C_w = {}^{B}$orbit of wx_0,

$$w_0C_w = w_0Bwx_0 = w_0Bw_0w_0wx_0$$
$$= B^-\underbrace{w_0w}\,x_0$$

So lets look at C_x and $w_0C_{w_0x}$

$\hookrightarrow Bxx_0$ $\quad\hookrightarrow B^-xx_0$ transverse!

U_x, $U^-_{w_0x}\subset G$ disjoint except e

project to C_x $w_0C_{w_0x}$ in G/B.

<u>Another approach</u>. $G/B \cong K/T$ $\quad$ K cpt real form $\quad$ $T \subset K$ max topo torus $(S^1)^n$.

Let $T^\vee = \mathrm{Hom}_{Grp}(T, S^1)$

$$\begin{tikzcd} K \arrow[r] \arrow[d] & EK \arrow[d] \\ \bullet \arrow[r] & BK \end{tikzcd} \rightsquigarrow \begin{tikzcd} K/T \arrow[r] \arrow[d] & EK/T = BT \arrow[d] \\ \bullet \arrow[r] & BK \end{tikzcd}$$

<u>Thm</u> pushout.

$$H^*(BT) \cong S(\mathfrak{t}^*) = \mathbb{R}[x_1, \dots, x_r]$$

$H^*(BK) \cong$ the W-invt part.